Demonstrating e using areas under curves

Allison Plant
WHBC, Wuhan, People's Republic of China
<allisonp@whbc2000.com>

Introduction

The number e is one of those fascinating numbers whose properties are of special interest to mathematicians. For example, the equation $e^{i\pi} + 1 = 0$ is said to be one of the most beautiful mathematical equations, in that it connects five of the most fundamental and interesting numerical constants: e, i, π, 1, and 0 (Devlin, 1997).

Recently Vahlas and Boukas (2007) provided a simple geometric construction of e. Certainly our approach is more naive than theirs; however, the ideas here are highly accessible to a diverse range of learners. This approach can be adapted to suit students with or without calculus knowledge.

The first two irrational numbers that we meet are, arguably, $\sqrt{2}$ and π. For both of these constants, students are often given a way to visualise the size of them: as the length of the diagonal of a unit square; and, as the length of the arc of a semicircle of radius 1, respectively. For the irrational number e, no such measure is usually given.

Our aim is to provide a method of introducing a visual concept of the number e. These ideas are suitable for secondary school and undergraduate tertiary students. The main concept involves areas under curves. Indeed, the number e is sometimes introduced as the base of the natural logarithm function. We suggest that if one follows the approach given here: to introduce e first; then the subsequent introduction of the natural logarithm function [$f(x) = \ln x$ or $f(x) = \log_e x$] is very natural.

The setting

Our goal, in each case, is to find the value a at which the area "under the curve" for the graph of the function $y = f(x)$ from $x = 1$ to $a = 1$, is 1. The preliminary examples are included to provide a conceptual setting. The cases listed here are designed in a "built up" way in order to help facilitate and develop the kind of situated cognitive experience that Brown, Collins and

Duguid (1989) argue is evidently successful for learning mathematics.

Case 1: f(x) =1

The graph of $y = 1$ is shown in Figure 1. The area under the curve from $x = 1$ to $x = a$ where $a > 1$ is determined by the area of a rectangle of height 1 and length $(a - 1)$, as illustrated in the figure. It is easy to see that if we require the area A under the curve to measure 1, then the value of a in this case must be 2; that is, putting $1 \times (a - 1) = 1$ yields $a = 2$.
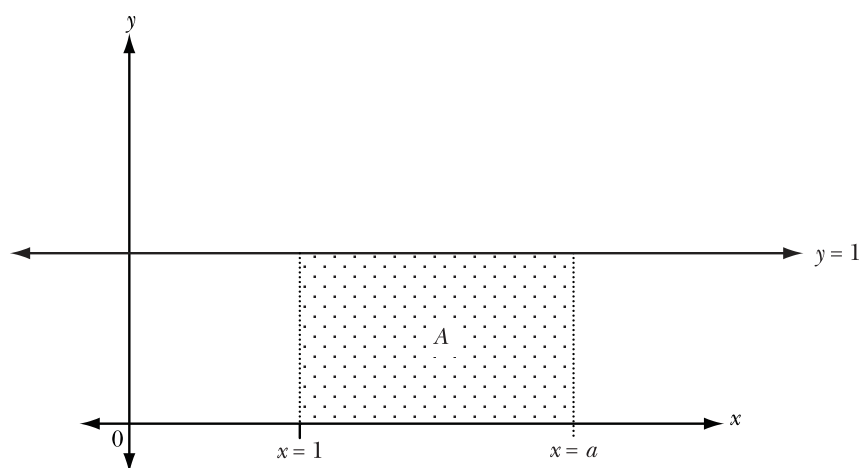

Figure 1. The area under the curve y = 1, from x = 1 to x = a, a > 1.

Case 2: f(x) = x

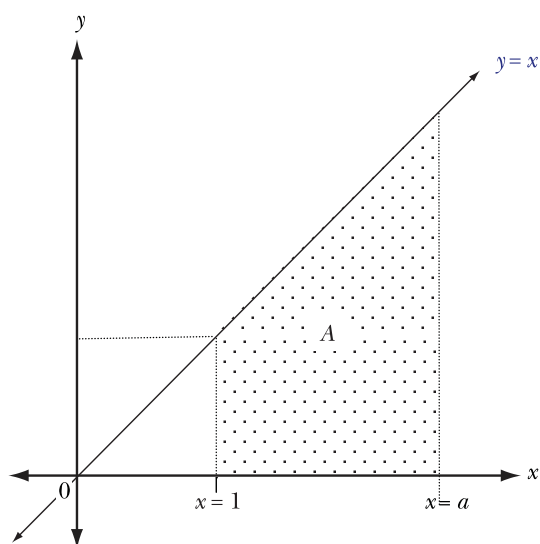In the next example, we let $y = x$. The appropriate graph is given in Figure 2.

Figure 2. The area under the curve y = x, from x = 1 to x = a.

In this case we are still readily able to calculate the area under the curve without needing calculus. The area can be broken into two simple shapes: a rectangle of length $(a - 1)$ and height 1, and a right-angled triangle of height and base both $(a - 1)$. So the total area A is given by

$$A = (a-1) + \frac{1}{2}(a-1)^2$$

This simplifies to give

$$A = \frac{1}{2}a^2 - \frac{1}{2}$$

Of course integrating $f(x) = x$ also yields the same result:

$$A = \int_1^a x\,dx = \left[\frac{x^2}{2}\right]_1^a = \frac{a^2}{2} - \frac{1}{2}$$

Now, since we want to find the value of a for which $A = 1$, we solve

$$1 = \frac{1}{2}a^2 - \frac{1}{2}$$

and find that $a = \sqrt{3}$.

Case 3: f(x) = x²

Our third example involves the function $f(x) = x^2$. Whilst it is easy to graph the function, finding the area under it without using calculus is not so trivial. Using integration, it follows easily that

$$A = \int_1^a x^2 dx = \left[\frac{x^3}{3}\right]_1^a = \frac{a^3}{3} - \frac{1}{3}$$

Again, we want the value of a for which $A = 1$, and so we solve

$$1 = \frac{1}{3}a^3 - \frac{1}{3}$$

for a and get the value

$$a = \sqrt[3]{4} = 2^{\frac{2}{3}}$$

For students without calculus, this case (with $f(x) = x^2$) can be omitted.

The interesting case

In the case with $y = \frac{1}{x}$ things become more revealing.

Firstly, since the exponent of x is -1, this function is the exception to the basic "student rules" of calculus for integrating polynomials. Secondly, the value we find for a turns out to be an interesting one (see Figure 3).

Remark
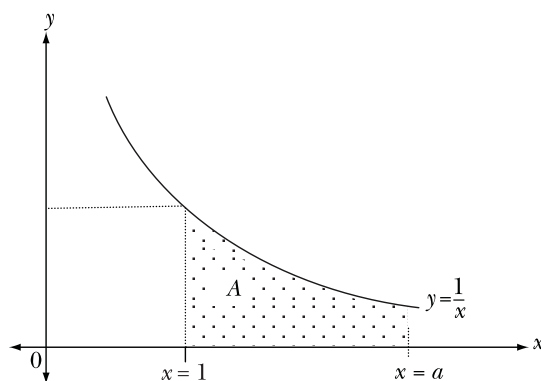The area under the curve $y = \frac{1}{x}$ from $x = 1$ to $x = a$ measures $A = 1$ when $a = e$.

Figure 3. Sketch of the curve $y = \dfrac{1}{x}$,
for which the area under the curve from x = 1 to x = a measures A = 1 when a = e.

We include a "proof" of our remark here to provide a complete account only. The proof is entirely straightforward only if $\ln x$ is defined. The point of this remark is that we could use this curve, following the preceding motivational setting, to *definitively* introduce the number *e*, and so in that case a proof for students would generally be omitted.

Proof

To find the area under the curve

$$y = \frac{1}{x}$$

from $x = 1$ to $x = a$, we integrate from $x = 1$ to $x = a$. Thus

$$A = \int_{1}^{a} \frac{1}{x}\, dx$$

$$= \left[\ln x\right]_{1}^{a}$$

$$= \ln a - \ln 1$$

$$= \ln a$$

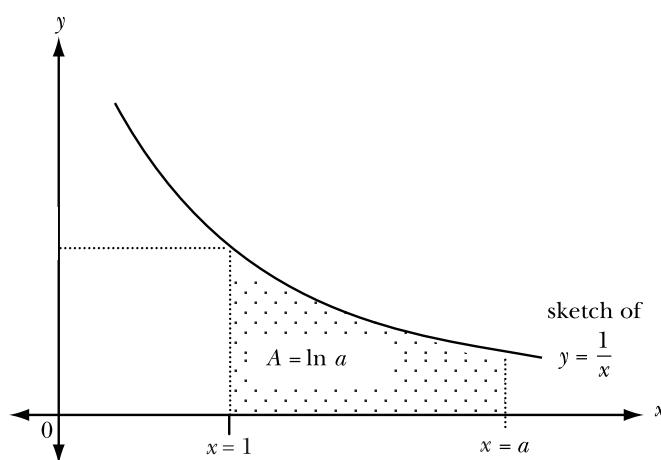Since we want A to equal 1, we have $\ln a = 1$, yielding $a = e$, as required.

Figure 4. The area under the curve $y = \dfrac{1}{x}$, *from x = 1 to x = a is called* $\ln a$.

Further remarks

Remark 1

We make the brief remark that the initial chain of examples ($y = x^0$; $y = x^1$; $y = x^2$) may highlight to students the need for developing techniques for finding the area under a curve: a motivation for integration.

Remark 2

If students have been introduced to the number e in the above mentioned way, then it becomes very easy to introduce the natural logarithm via the usual approach as a measure of the area under the curve

$$y = \frac{1}{x}$$

Further to this, some students may be interested in a visual idea of the measure of e^2 is, or perhaps even the measure of e^π. Indeed the number e raised to any power r is found along the x-axis at the value $x = a$ which yields an area, under the curve

$$y = \frac{1}{x}$$

from $x = 1$ to $x = a$, of the same measure as the power r. As an example, for e^2 we need the area under the curve from $x = 1$ to $x = a$ to equal 2. In other words, the area under the curve will equal 2 when we evaluate from $x = 1$ to $x = e^2$. In this way we are able to introduce the neat idea of one irrational number raised to the power of another irrational number. Indeed, Figure 5 illustrates the measure of $e^{\sqrt{2}}$.
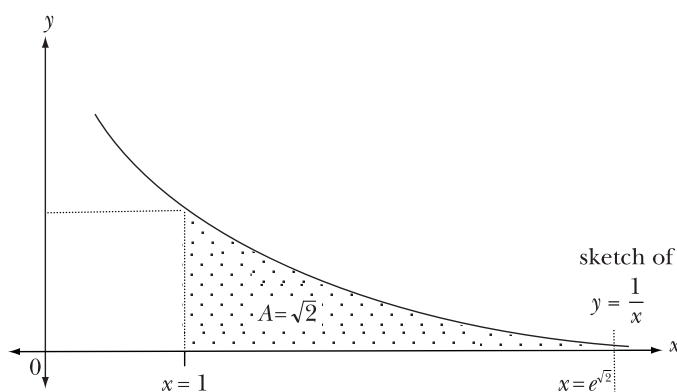

Figure 5. The area under the curve $y = \frac{1}{x}$, from $x = 1$ to $x = e^{\sqrt{2}}$ is $\sqrt{2}$.

Remark 3

It is important in the teaching of mathematics to guide our students toward abstraction and generalisation when possible. This better prepares them to apply their skills to a broader variety of real-world situations (Laurillard, 2002). To this end, we have the opportunity to develop a proposition which describes the pattern unfolding in the previous cases. This inclusion is intended only for students with calculus. Before proceeding to the proposition, it will be useful to gain more evidence of a pattern by first introducing a further case, $f(x) = x^3$. The details of this case are left to the reader.

Proposition

For any monomial of the form $f(x) = x^n$ (where n is a natural number) the value for a which yields an area under the curve of measure 1 from $x = 1$ to $x = a$ is given by

$$a = \sqrt[(n+1)]{(n+2)}$$

Proof

The area A under the curve $f(x) = x^n$ from $x = 1$ to $x = a$ is determined by

$$A = \int_1^a x^n dx = \left[\frac{x^{(n+1)}}{(n+1)} \right]_1^a = \frac{a^{(n+1)}}{(n+1)} - \frac{1}{(n+1)}$$

Since $A = 1$, we have

$$\frac{a^{(n+1)}}{(n+1)} - \frac{1}{(n+1)} = 1$$

Multiplying the common denominator through and isolating a yields $a^{(n+1)} = n + 2$, from which the result immediately follows.

References

Brown, J. S., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher, 18*(1), 32–42.

Devlin, K. (1997). *Mathematics: The science of patterns.* New York: W. H. Freeman.

Laurillard, D. (2002). *Rethinking university teaching: A framework for the effective use of learning technologies (2nd ed.).* London: Routledge Falmer.

Valahas, T. M. & Boukas, A. (2007). A simple geometric construction of *e. Australian Senior Mathematics Journal, 21*(2), 29–30.